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# Renormalization Scheme Dependence and the Problem of Theoretical Uncertainties in Next-Next-to-Leading Order QCD Predictions \*

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## Abstract

Renormalization scheme uncertainties in the next-next-to-leading order QCD predictions are discussed. To obtain an estimate of these uncertainties it is proposed to compare predictions in all schemes that do not have unnaturally large expansion coefficients. A concrete prescription for eliminating the unnatural schemes is given, based on the requirement that large cancellations in the expression for the characteristic renormalization scheme invariant should be avoided. As an example the QCD corrections to the Bjorken sum rule are considered. The importance of the next-next-to-leading order corrections for a proper evaluation of perturbative QCD predictions is emphasized.

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In the process of evaluation of perturbative QCD predictions one inevitably encounters the problem of renormalization scheme (RS) dependence. As is well known, despite the fact that final predictions of a quantum field theory for physical quantities should be independent of choice of the renormalization scheme, the results obtained with the truncated perturbation series may be numerically different in different schemes. Although the difference is formally of higher order in the coupling constant, numerically it may be significant in the phenomenological analysis. This effect complicates somewhat comparison of the theoretical predictions with the experimental data. There has been a lively discussion on the problem of a proper choice of the RS [1, 2, 3, 4, 5, 6].

However, the problem of a proper choice of the RS is only one of the aspects of the RS dependence effect. In fact, the RS dependence of perturbative predictions, instead of being an annoying complication, may also become a source of an important information. The idea is to estimate the strength of the RS dependence, and to use it as a measure of reliability of perturbation expansion. Indeed, we expect the perturbative expression to be less reliable when the expansion coefficients are large and/or when the coupling constant is not very small. These are precisely the conditions under which we expect also a strong RS dependence of perturbative predictions.

A procedure that is frequently used to estimate the strength of the RS dependence consists of calculating the variation of predictions when the renormalization scale is changed in some reasonable range. Unfortunately, in the next-next-to-leading order (NNLO) approximation the freedom of choice of RS is parametrized by *two* independent parameters, so that varying the renormalization scale we scan only a small subset of all the available schemes. Also, as has been discussed for example in [7], the concept of a “reasonable” renormalization scale is not well defined.

A more general approach was formulated in [7], in which full freedom of choice of the RS is taken into account. This approach is based on the observation that in NNLO one may select acceptable schemes using the expression for a RS invariant combination of the expansion coefficients. In this note the ideas presented in [7] are further developed. A precise mathematical formulation of the constraint eliminating the unnatural schemes is given, which allows for a quantitative estimate of the strength of RS dependence for any physical observable for which the NNLO corrections are known. The QCD corrections to the Bjorken sum rule for polarized structure functions are discussed as an example. The results

obtained for other observables are also briefly summarized [8].

Let us consider a NNLO expression for a physical quantity  $R$  depending on a single energy variable  $P$ :

$$R^{(2)}(P) = a(\mu)[1 + r_1(\mu/P)a(\mu) + r_2(\mu/P)a^2(\mu)], \quad (1)$$

where  $\mu$  denotes the renormalization scale,  $a(\mu) = g^2(\mu)/(4\pi^2)$  denotes the running coupling constant that satisfies the NNLO renormalization group equation:

$$\mu \frac{da}{d\mu} = -b a^2[1 + c_1 a + c_2 a^2], \quad (2)$$

and where

$$r_1(\mu/P) = r_1^{(0)} + b \ln(\mu/P), \quad (3)$$

$$r_2(\mu/P) = r_2^{(0)} + (c_1 + 2r_1^{(0)})b \ln(\mu/P) + (b \ln(\mu/P))^2, \quad (4)$$

The coefficients  $r_i^{(0)}$  are independent of  $\mu$ . The commonly used renormalization group improved expression for  $R$  is obtained by assuming that  $\mu = kP$ , where  $k$  is a constant. In the following it is assumed, that the quark mass effects are approximated by a step-function energy dependence of the number of active quark flavors.

The results of perturbative calculations are usually expressed in the modified minimal subtraction ( $\overline{MS}$ ) scheme [9]. Other choices of RS are possible, corresponding to different choices of the finite parts of renormalization constants. Results in various schemes are related by a finite renormalization, which in our approximation is equivalent to a redefinition of the coupling constant:

$$a_{\overline{MS}}(\mu) = a(\mu)[1 + A_1 a(\mu) + A_2 a^2(\mu)]. \quad (5)$$

The constants  $A_1$  and  $A_2$  may in principle be arbitrary. The expansion coefficients  $r_1$ ,  $r_2$  and  $c_2$  depend on the choice of RS. Also the parameter  $\Lambda$  does depend on the RS [10]. The relevant formulas describing this dependence are collected for example in [7]. However, in NNLO there exists an RS independent combination of the expansion coefficients [2, 11, 12]:

$$\rho_2 = c_2 + r_2 - c_1 r_1 - r_1^2. \quad (6)$$

As was pointed out in [7], a convenient description of the RS dependence of  $R^{(2)}$  is obtained when  $r_1$  and  $c_2$  are chosen as independent parameters characterizing the choice of RS. The

parameter  $r_1$  controls the degree of freedom in the approximants which is equivalent to the freedom of choice of the renormalization scale  $\mu$ , and  $c_2$  controls – in the terminology of Stevenson [2] – the dependence on the choice of the renormalization convention. With this parametrization the NNLO RG-improved expression for  $R$  may be written in the form [7]:

$$R^{(2)}(P/\Lambda_{\overline{MS}}; r_1, c_2) = a(P)[1 + r_1 a(P) + r_2(r_1, c_2) a^2(P)], \quad (7)$$

where  $r_2$  is expressed as a function of  $r_1$  and  $c_2$  using the Eq. 6. The numerical value of the running coupling constant is obtained from the implicit equation:

$$b \ln\left(\frac{P}{\Lambda_{\overline{MS}}}\right) = r_1^{\overline{MS}} - r_1 + \Phi^{(2)}(a, c_2), \quad (8)$$

where

$$\Phi^{(2)}(a, c_2) = c_1 \ln\left(\frac{b}{2c_1}\right) + \frac{1}{a} + c_1 \ln(c_1 a) + O(a), \quad (9)$$

which is obtained by integrating the Eq. 2 with an appropriate boundary condition. The explicit form of  $\Phi(a, c_2)$  may be found for example in [13]. The presence of  $\Lambda_{\overline{MS}}$  in the Eq. 8 is a result of taking explicitly into account the one loop Celmaster-Gonsalves relation between  $\Lambda$  parameters in different schemes [10], which is valid to all orders of perturbation expansion.

The fact, that numerical value of the predictions depends on the choice of RS, stimulated the interest in prescriptions, which pick up schemes distinguished by some arguments of physical or mathematical character. Several such prescriptions for selecting a “good” RS have been discussed, like the principle of Fastest Apparent Convergence (FAC) [1], the Principle of Minimal Sensitivity (PMS) [2], the momentum subtraction prescription [3, 4], and other [5]. However, regardless of our choice of the optimal scheme, there is always a continuum of schemes which are close to the one preferred by us. Since any choice of a preferred RS may have only a heuristic motivation, the predictions obtained in schemes close to the optimal scheme also must be somehow taken into account. A natural way to accomplish this is to supplement the prediction in a preferred scheme by the estimate of the strength of the scheme dependence.

In order to obtain a meaningful estimate of the strength of RS dependence one should calculate variation of predictions over all *a priori* acceptable schemes. To this end we must provide some well motivated criteria for selecting the admissible scheme. The use of some

selection criteria cannot be avoided, because by taking variation of predictions over *all* possible schemes we would obtain a meaningless result. The reason for this is obvious – clearly one may make a bad choice of scheme, in which the perturbation expansion could have unnaturally large expansion coefficients, and one may artificially obtain very large higher order corrections and very strong RS dependence, without any physical significance. As was pointed out in [7], in order to eliminate unnatural schemes one may use the expression for the RS invariant  $\rho_2$ . This invariant appears as an NNLO expansion coefficient in the manifestly RS-invariant evolution equation for a physical quantity [11, 12, 14], and therefore it may be taken as a natural RS-independent characterization of the magnitude of the NNLO corrections. The observation made in [7] was that one should avoid using schemes that introduce large cancellations in the expression for  $\rho_2$ .

To make this condition more concrete let us introduce a function which describes the degree of cancellation in the expression for  $\rho_2$  in a quantitative way:

$$\sigma_2(r_1, r_2, c_2) = |c_2| + |r_2| + c_1|r_1| + r_1^2. \quad (10)$$

The requirement, that the contributions to  $\rho_2$  should not be too large compared to the magnitude of  $\rho_2$  itself, may be now written in the form:

$$\sigma_2(r_1, r_2, c_2) \leq l|\rho_2|, \quad (11)$$

where  $l \geq 1$  is some constant. The constant  $l$  determines how strong cancellations in the expression for  $\rho_2$  we want to allow. For arbitrary  $l$  it is straightforward to obtain an explicit description of the region of allowed parameters in the  $(r_1, c_2)$  plane. In the case  $\rho_2 > c_1^2/4$ , which is what will be needed for the Bjorken sum rule, let us define:

$$r_1^{min} = -\sqrt{\rho_2(l-1)/2}, \quad (12)$$

$$r_1^{max} = [-c_1 + \sqrt{c_1^2 + 2(l-1)\rho_2}]/2, \quad (13)$$

$$c_2^{min} = -\rho_2(l-1)/2, \quad (14)$$

$$c_2^{max} = \rho_2(l+1)/2, \quad (15)$$

$$c_2^{int} = c_1 r_1^{min} + c_2^{max}. \quad (16)$$

For  $c_2 > 0$  the region of allowed parameters is bounded from above by the line joining the points  $(r_1^{min}, 0)$ ,  $(r_1^{min}, c_2^{int})$ ,  $(0, c_2^{max})$ ,  $(r_1^{max}, c_2^{max})$ ,  $(r_1^{max}, 0)$ . For  $c_2 < 0$  the region of allowed

parameters is bounded from below by the lines:

$$c_2(r_1) = r_1^2 + c_2^{min} \quad for \quad r_1^{min} \leq r_1 \leq 0, \quad (17)$$

$$c_2(r_1) = r_1^2 + c_1 r_1 + c_2^{min} \quad for \quad 0 \leq r_1 \leq r_1^{max}. \quad (18)$$

It is interesting that the FAC scheme, corresponding to  $r_1 = 0$ ,  $c_2 = \rho_2$ , always belongs to the allowed region. We also see, that for  $\rho_2 = 0$  – i.e. null NNLO correction – the only scheme satisfying Eq. 11 is  $r_1 = 0$ ,  $c_2 = 0$  (and of course  $r_2 = 0$ ).

For a final specification of the allowed region in the  $(r_1, c_2)$  plane we have to choose some value of  $l$  in the Eq. 11. To make a meaningful choice let us observe, that it is a rather natural requirement to have the PMS scheme in the allowed region. The PMS scheme parameters are determined by the condition:

$$\frac{\partial R^{(2)}}{\partial r_1} = 0, \quad \frac{\partial R^{(2)}}{\partial c_2} = 0. \quad (19)$$

An approximate solution of these equations has the form [15]:

$$r_1^{PMS} = 0(a^{PMS}), \quad c_2^{PMS} = \frac{3}{2}\rho_2 + 0(a^{PMS}). \quad (20)$$

It is easy to see that taking  $l = 2$  we would have the PMS scheme right at the boundary of the allowed region in the  $(r_1, c_2)$  plane.

Therefore, in order to have an estimate of the strength of the scheme dependence in NNLO we propose to calculate the variation of predictions using Eq. 7 and Eq. 8, when the scheme parameters  $r_1$  and  $c_2$  are changed within the region satisfying the Eq. 11 with  $l = 2$ .

Let us now consider the QCD correction  $R_{Bj}$  to the Bjorken sum rule for the polarized structure functions in the deep-inelastic scattering [16]:

$$\int_0^1 dx [g_1^p(x, Q^2) - g_1^n(x, Q^2)] = \frac{1}{6} \frac{g_A}{g_V} [1 - R_{Bj}(Q^2)]. \quad (21)$$

This sum rule has been recently a subject of intense experimental [17, 18] and theoretical [19, 20] investigation. The QCD perturbation expansion for  $R_{Bj}$  has the form of Eq. 1, with  $P^2 = Q^2$ , where the expansion coefficients in the  $\overline{MS}$  scheme with four quark flavors are given by [21, 19]:

$$r_1^{\overline{MS}} = 3.25 \quad r_2^{\overline{MS}} = 13.8503. \quad (22)$$

Together with  $c_1 = 1.54$ ,  $c_2^{\overline{MS}} = 3.0476$  this implies  $\rho_2 = 1.3304$  ( $b = 25/6$ ). In Figure 1 we show  $R_{Bj}^{(2)}$  as a function of  $r_1$  and  $c_2$  for  $P/\Lambda_{\overline{MS}} = 8.6$  (this corresponds to  $\langle Q^2 \rangle = 5 \text{ GeV}^2$  – as used by the SMC Collaboration to present the combined experimental data [18] – and  $\Lambda_{\overline{MS}}^{(4)} = 0.26 \text{ GeV}$  as preferred by the Particle Data Group [22]), together with the region of allowed values for the scheme parameters. We see that  $R_{Bj}^{(2)}$  has indeed a saddle point, representing prediction in the PMS scheme, close to the  $l = 2$  allowed region. We have:

$$R_{Bj}^{PMS}(8.6) = R_{Bj}^{(2)}(8.6; 0.0852, 2.1302) = 0.126525. \quad (23)$$

The estimate of the strength of the scheme dependence is obtained by comparing the predictions inside the allowed region. We find that the extremal values are attained at the boundary of the allowed region:

$$R_{Bj}^{max}(8.6) = R_{Bj}^{(2)}(8.6; -0.668, -0.219) = 0.12697, \quad (24)$$

$$R_{Bj}^{min}(8.6) = R_{Bj}^{(2)}(8.6; 0.352, 0) = 0.12597. \quad (25)$$

For comparison we also show in Figure 1 the range of scheme parameters satisfying the constraint given by Eq. 11 with  $l = 3$ , in which case a small neighbourhood of the PMS parameters is taken into account. In this case we obtain:

$$R_{Bj}^{max}(8.6) = R_{Bj}^{(2)}(8.6; -0.838, -0.628) = 0.12722, \quad (26)$$

$$R_{Bj}^{min}(8.6) = R_{Bj}^{(2)}(8.6; 0.617, 0) = 0.12551. \quad (27)$$

In Figure 2 we show the prediction for  $R_{Bj}^{(2)}$  as a function of  $P/\Lambda_{\overline{MS}}$  near the point  $P/\Lambda_{\overline{MS}} = 8.6$ . This figure shows how the uncertainty of the predictions arising from the RS dependence may affect the fit of  $\Lambda_{\overline{MS}}$  to possible experimental results. We see that the obtained scheme dependence appears to be very small. In fact it is much smaller than the uncertainty arising from the estimated error in the fitted value of  $\Lambda_{\overline{MS}}$  – the Particle Data Group gives  $\Lambda_{\overline{MS}}^{(4)} = 0.316 - 0.214 \text{ GeV}$ , which for  $Q^2 = 5 \text{ GeV}^2$  gives variation of  $Q^2/\Lambda^2$  in the range  $(7.08)^2 - (10.45)^2$ . The scheme dependence uncertainty is also much smaller than the experimental errors in the presently available experimental data [17, 18]. This shows that the  $n_f = 4$  NNLO formula for the QCD corrections to the polarized Bjorken sum rule

seems to be a potentially good source of precise information on  $\Lambda_{\overline{MS}}$  once the experimental accuracy is improved.

Let us also note that the  $\overline{MS}$  coefficients lie outside the allowed region indicated in Figure 1, i.e. in our approach the  $\overline{MS}$  scheme has to be considered as a scheme with unnaturally large expansion coefficients. Also,  $R_{Bj}^{\overline{MS}}(8.6) = 0.11969$  lies outside the range of values obtained in our estimate of scheme dependence uncertainties. However, the difference between  $R_{Bj}^{\overline{MS}}$  and  $R_{Bj}^{PM\overline{S}}$  is still rather small for  $Q^2/\Lambda_{\overline{MS}}^2$  close to  $(8.6)^2$ .

For the sake of comparison it is interesting to consider the RS dependence of the NNLO expression for  $R_{Bj}$  for  $n_f = 3$ . Taking three quark flavors we have:

$$r_1^{\overline{MS}} = 3.5833 \quad r_2^{\overline{MS}} = 20.2153, \quad (28)$$

which together with  $c_1 = 16/9$  and  $c_2^{\overline{MS}} = 4.471$  implies  $\rho_2 = 5.4757$  ( $b = 9/2$ ). This is a significantly larger value than for four flavors. The dependence of  $R_{Bj}^{(2)}$  on  $r_1$  and  $c_2$  for  $n_f = 3$  for  $P/\Lambda_{\overline{MS}} = 8.6$  is shown in Figure 3. We find:

$$R_{Bj}^{PM\overline{S}} = R_{Bj}^{(2)}(8.6; 0.33, 8.83) = 0.121634. \quad (29)$$

Repeating the procedure used for  $n_f = 4$  we obtain for the  $l = 2$  allowed region:

$$R_{Bj}^{max}(8.6) = R_{Bj}^{(2)}(8.6; -1.655, 0) = 0.12748, \quad (30)$$

$$R_{Bj}^{min} = R_{Bj}^{(2)}(0.989, 0) = 0.11757. \quad (31)$$

For the  $l = 3$  allowed region we find:

$$R_{Bj}^{max}(8.6) = R_{Bj}^{(2)}(8.6; -2.340, 0) = 0.13412, \quad (32)$$

$$R_{Bj}^{min}(8.6) = R_{Bj}^{(2)}(8.6; -2.340, 6.791) = 0.10683. \quad (33)$$

Note that also in this case the  $\overline{MS}$  coefficients lie outside the allowed range for scheme parameters. We have  $R_{Bj}^{\overline{MS}} = 0.11102$ , which is close to the lower limit of values obtained by varying over the  $l = 3$  allowed region. We see that the theoretical uncertainty in the  $n_f = 3$  formula is larger than in the  $n_f = 4$  formula. This is a result of stronger dependence of  $R^{(2)}$  on the scheme parameters in this case, and of the larger range of parameters which



may be considered to be natural. Using our method we may now quantitatively compare reliability of the  $n_f = 3$  and  $n_f = 4$  predictions for  $R_{Bj}$ . Concerning the energy dependence of the theoretical uncertainties in this case let us note, that for lower values of  $Q^2/\Lambda^2$  the RS dependence of the three-flavor expression, estimated according to our prescription, is considerably larger than that found above. The problem of the low energy behavior of  $R_{Bj}^{(2)}$  and a possible method for reducing scheme dependence at low energies would be discussed in a separate note [23].

We conclude this note with several remarks.

1. It has to be emphasized that our estimate of the strength of the RS dependence does not provide an error estimate in the mathematical sense, i.e. there is no theorem that would guarantee that the true result lies within the obtained range of variation for the prediction. The method described above provides only a consistent framework for making an unbiased educated guess about possible theoretical precision of the prediction. The method avoids unnecessary constraints on the considered renormalization schemes, it eliminates from analysis the unnatural schemes, and it is sensitive to the magnitude of the NNLO correction via dependence on  $\rho_2$ . The estimate of the scheme dependence uncertainty obtained in this way should be particularly useful for a quantitative comparison of reliability of perturbative predictions for different physical quantities, evaluated at different energies. In fact one may reasonably expect that thus obtained *relative* estimate of the reliability of perturbative predictions would be insensitive to certain freedom in choosing  $l$  in the Eq. 11 and to possible variations in the choice of the function  $\sigma_2$ . The proposed method should be also useful in determining the regions of applicability of perturbation expansion. Indeed, since we have taken great care to eliminate schemes with unnaturally large expansion coefficients, in order to avoid introducing any spurious RS dependence, then any large variation of predictions obtained according to our method would be an unambiguous sign of a limited applicability of the NNLO expression in the considered energy range.

2. The RS invariant  $\rho_2$  plays a fundamental role in our approach. However, Stevenson in his analysis of RS dependence [2] used an expression for the NNLO invariant that differs from the one used here by a constant. We think that the expression used in this note is more natural for characterization of the magnitude of NNLO corrections because it appears in the RS invariant evolution equation for a physical quantity [11, 12, 14]. To emphasize the RS independent character of the evolution equation obtained in [11, 12, 14] let us recall

the argument indicated in [24]. Let us take  $R^{(2)}$  as given by Eq. 1 in *arbitrary* scheme, differentiate over  $P$ , then express  $a(P)$  in terms of  $R^{(2)}$ , and finally express  $P(dR^{(2)}/dP)$  as a series expansion in  $R^{(2)}$ . We obtain:

$$P \frac{dR^{(2)}}{dP} = -b(R^{(2)})^2 [1 + c_1 R^{(2)} + \rho_2 (R^{(2)})^2 + \sum_{k=3}^{\infty} \overline{\rho}_k (R^{(2)})^k]. \quad (34)$$

The first three expansion coefficients in this equation are RS independent and coincide with the coefficients obtained in [11, 12, 14]. Higher order expansion coefficients in this equation are scheme dependent, which is a result of the fact that we have only used NNLO approximation for a physical quantity. (If higher order approximant would be taken, then more coefficients in the above equation would become independent of the choice of RS.) This shows, that the equation obtained in [11, 12, 14] is indeed RS independent, and that it is not a result of a particular choice of the scheme.

Another argument supporting our choice is the fact that  $\rho_2$  naturally appears in the leading order solution of the PMS equations (Eq. 20).

3. The fact that we base our method on the existence of the RS invariant  $\rho_2$  has an obvious consequence, that it cannot be applied to quantities for which only next-to-leading (NLO) corrections are known. This should not be surprising however, because the NLO approximation has a rather special status in QCD. In fact, the NLO correction allows only to set the proper energy scale for the prediction. For example, using the FAC scheme we obtain in NLO an *identical* energy dependence for *all* possible physical observables depending on one energy variable, except for a shift on a logarithmic energy scale. It is only at NNLO that physical quantities receive process-specific corrections. It is therefore natural that only at NNLO and beyond one may test reliability of the perturbation expansion. This indicates a fundamental importance of the NNLO calculations for a proper comparison of theoretical predictions with experimental data. Let us also note that it is straightforward to extend our approach to the case of four-loop or even higher order corrections if such calculations would ever be done.

4. Let us discuss some other methods of estimating the reliability of perturbative predictions. A simplest alternative is to compare NLO and NNLO predictions, taking the difference as an estimate of the theoretical uncertainty. Unfortunately, it is obvious that such an estimate would be strongly dependent on the choice of scheme. One may also try to guess

the magnitude of higher order corrections. Again, the problem here is the RS dependence of such an approach. Note also that the asymptotic expression for higher order corrections obtained in [25], which has been sometimes used to evaluate the theoretical uncertainty, is known to be erroneous [26]. Finally, in [27] it is proposed to obtain theoretical uncertainty by comparing predictions obtained in the PMS, FAC and  $\overline{MS}$  schemes. Unfortunately, this method also may give misleading results, since the predictions in the PMS and FAC approach are known to lie rather close to each other regardless of the magnitude of the NNLO correction, and the choice of  $\overline{MS}$  in such procedure is completely arbitrary, without any theoretical or phenomenological motivation. In fact, as was shown above, comparing PMS and  $\overline{MS}$  we may obtain *larger* uncertainty than that suggested by our method.

5. The method described in this note has been applied to other quantities for which the NNLO expansion is known [23, 28, 29].

The results for the Gross-Llewellyn-Smith sum rule are similar to those obtained above for the Bjorken sum rule. The NNLO expression for  $n_f = 4$  appears to have small scheme uncertainty, while the  $N_f = 3$  expression seems to be less precise, particularly at lower energies.

In the case of the QCD corrections to the tau lepton decay we confirm the observations made in [7]. By varying predictions over the set of schemes allowed by the constraint given by Eq. 11 we find that the frequently used NNLO perturbative expression is unstable against the change of RS. (In [7] the condition on the allowed schemes was not specified in a quantitative way.) However, if one takes a contour integral representation for  $R_\tau$ , one obtains a stable result [28], with theoretical uncertainty of the same order as the present experimental uncertainty. Neglecting nonperturbative contributions for simplicity and fitting  $\Lambda_{\overline{MS}}$  to the experimental central value  $R_\tau^{exp} = 0.2$  we obtain  $0.396 \text{ GeV}$  in the generalized PMS method, and  $0.386 - 0.415 \text{ GeV}$  variation for the  $l = 2$  allowed region for the scheme parameters.

Application of our method to the case of the  $n_f = 5$  QCD corrections to the  $e^+e^-$  annihilation into hadrons reveals a surprisingly large RS dependence, despite rather high characteristic energy scale. The large scheme uncertainty is a consequence of a large value of the RS invariant  $\rho_2$ . However, a closer analysis shows that large value of this invariant is due mainly to the effect of analytic continuation from the spacelike to timelike momenta. A

proper treatment of the analytic continuation results in a considerable improvement of the stability of the predictions with respect to change of the RS [29]. Similar remarks apply to the case of the QCD corrections to the  $Z$ -boson decay rate to hadrons, which is however more complicated because of the presence of mass dependent corrections.

Summarizing, we may say, that a method has been proposed for analyzing the uncertainties in the NNLO predictions, introduced by arbitrariness in the choice of the renormalization scheme. This method is based on a condition that eliminates schemes which have expansion coefficients giving unnaturally large cancellations in the expression for the characteristic NNLO RS invariant. Application of the method has been illustrated using as an example the QCD corrections to the Bjorken sum rule for polarized structure functions. It was found that for four quark flavors these corrections have small scheme dependence uncertainty. Our considerations show the fundamental importance of the NNLO corrections for a proper evaluation of the perturbative QCD predictions.

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## Figure Captions

Fig. 1. Contour plot of the NNLO prediction for  $R_{Bj}$  as a function of  $r_1$  and  $c_2$ , for  $n_f = 4$  and  $Q^2/\Lambda_{\overline{MS}}^2 = (8.6)^2$ . The region of values of the scheme parameters satisfying the constraint given by Eq. 11 is indicated for  $l = 2$  (smaller region) and for  $l = 3$ . The dashed line corresponds to  $R_{Bj}^{(2)} = 0.126526$ .

Fig. 2. The NNLO prediction for  $R_{Bj}$  as a function of  $P/\Lambda_{\overline{MS}}$  ( $P^2 = Q^2$ ). The thick broken line in the middle denotes the PMS prediction. The thin solid lines denote the variation of the predictions when  $r_1$  and  $c_2$  are changed within a region defined by Eq. 11 with  $l = 2$ . Thin broken lines correspond to  $l = 3$ .

Fig. 3. Same as in Fig. 1 but for  $n_f = 3$ . The dashed line corresponds to  $R_{Bj}^{(2)} = 0.121635$ .

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